THE PROBLEM OF THERMAL CONDUCTIVITY FOR A FINITE RATE OF HEAT PROPAGATION

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A solution is given for the problem of heat propagation along semibounded and bounded rods with a constant-power heat source, with and without consideration of the transfer of heat from the side surface for a finite rate of heat propagation.

There are numerous solutions [1-3] for the problem relating to the propagation of heat in bodies or in a system of bodies under various boundary conditions. As demonstrated in [4], in principle any such solution may be used to develop a method for the determination of the thermophysical characteristics of various materials under the conditions of the nonsteady problem, which makes it possible to undertake an integrated study. Common to this class of problems is the assumption that the rate of heat propagation is infinitely large, as a consequence of which the basic heat-conduction equation is written as

$$\frac{\partial T}{\partial \tau} = a\nabla^2 T, \tag{1}$$

instead of the more exact hyperbolic heat-conduction equation [2]

$$\frac{\partial T}{\partial \tau} + \tau_r \frac{\partial^2 T}{\partial \tau^2} = a\nabla^2 T, \tag{2}$$

which takes into consideration the finiteness of the rate of heat propagation.

Such an approach, at normal temperatures and pressures (for gases), is justified by the fact that under these conditions the relaxation time τ_r is on the order of 10^{-9}-10^{-11} sec and, consequently, the effects associated with the finiteness of the rate of heat propagation will not markedly affect the experimental results, because contemporary techniques are not up to the task of spotting these.

We find a rather unique situation in the region of low and superlow temperatures and great rarefaction (for gases). Since the relaxation time $\tau \sim \Lambda/U$, with a change in temperature it varies as $\sim T^{-3}$ [5], while with a drop in pressure it varies as $\sim p^{-1}$ [6]. Under these conditions the thermal diffusivity a remains either constant (when $\lambda \sim T^3$ and $c \sim T^3$) or it varies in proportion to T^{-2} (when $\lambda \sim T$ and $c \sim T^3$).

All of this leads to the fact that with a drop in temperature and pressure the second term in the left-hand member of (2) becomes commensurate in terms of magnitude with the right-hand member. Consequently, in solving heat-conduction problems in the region of low temperatures and pressures we must use (2), whose particular solutions may serve as the basis for the development of a method of integrated determination of the thermophysical characteristics for various materials in the given range of parameter variations.

Let us consider the problem of the heat conduction of a semibounded rod with a heat-insulated surface in the case of a finite rate of heat propagation. The basic heat-conduction equation (2) is then written as

$$\frac{\partial T(x,\tau)}{\partial \tau} + \tau_r \frac{\partial^2 T(x,\tau)}{\partial \tau^2} = a\frac{\partial^2 T(x,\tau)}{\partial x^2}, \tag{3}$$

where we assume that τ_r and a are independent of temperature. The boundary conditions of the problem are

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$$q = -\lambda \left(\frac{\partial T}{\partial x}\right)_{x=0} = \text{const}, \quad (4)$$

$$T(x, 0) = T(\infty, \tau) = T_0 = \text{const}. \quad (5)$$

Moreover, for (2) to be a transport equation and to satisfy the law for the conservation of energy, the solution must satisfy the condition*

$$\left(\frac{\partial T}{\partial \tau}\right)_{\tau=0} = 0. \quad (6)$$

We seek the solution through use of the Laplace transform. Applying the Laplace transform, we obtain the solution in the following form:

$$T(x, \tau) - T_0 = 0 \text{ when } \tau < \tau_d = x\sqrt{\frac{\tau_r}{a}}, \quad (7)$$

$$T(x, \tau) - T_0 = \frac{q\sqrt{a}}{\lambda\sqrt{\tau_r}} \int_{\tau_d}^{\tau} e^{-\frac{t}{2\tau_r}} I_0\left(\frac{\sqrt{t^2 - \tau_d^2}}{2\tau_r}\right) dt \quad (8)$$

when $\tau > \tau_d$

As we can see, the presence of a finite rate of heat propagation determines the existence of a fully determined value for the delay time τ_d, and it is only on elapse of this time that the temperature of the body begins to change at a given point.

When the surface of the rod is not heat-insulated and, bearing in mind the combined transfer of heat by convection and radiation, according to [2], in terms of an arbitrary coefficient $\alpha = \alpha_{conv} + \sigma^* b(T)$, where $\sigma^* = c_0\varepsilon$ is the reduced coefficient of radiation, and b(T) is some function dependent on the temperatures of the rod and the medium, the basic equation (2) can be written as follows:

$$\frac{\partial T(x, \tau)}{\partial \tau} + \tau_r \frac{\partial^2 T(x, \tau)}{\partial \tau^2} = a \frac{\partial^2 T(x, \tau)}{\partial x^2} - \frac{\alpha}{c_r \rho h}[T(x, \tau) - T_0]. \quad (9)$$

The boundary conditions (4)-(6) remain as before. The solution for (9) will have the form

$$T(x, \tau) - T_0 = 0 \text{ when } \tau < \tau_d, \quad (10)$$

$$T(x, \tau) - T_0 = \frac{q}{\lambda}\sqrt{\frac{a}{\tau_r}} \int_{\tau_d}^{\tau} e^{-\frac{t}{2\tau_r}} I_0\left(\frac{\sqrt{1 - 4H\tau_r}}{2\tau_r}\sqrt{t^2 - \tau_d^2}\right) dt. \quad (11)$$

Here, as before, $I_0(z)$ is a zero-order Bessel function of imaginary argument. Comparing solutions (8) and (10), we see that the latter differs only in the presence of the factor $\sqrt{1 - 4H\tau_r}$ in the argument of the integrand. Since the argument must be a real number, we find the condition

$$H < \frac{1}{4\tau_r} \quad \text{or} \quad \tau_r < \tau_l/4, \quad (12)$$

and if this condition is satisfied we will have a nonsteady variation in the temperature of the rod as the latter is heated. It should be noted that under ordinary conditions, because of the small magnitude of τ_r, this condition, as a rule, is always satisfied.

Having calculated the approximate values of the integrals in (8) and (11) and assuming x = 0, for large time intervals ($\tau \to \infty$) we can find that in the absence of heat transfer

$$[T(0, \tau) - T_0]_{\tau \to \infty} \to \frac{2}{\sqrt{\pi}} \frac{q\sqrt{a\tau}}{\lambda},$$

which is in agreement with the solution from [2], while in the case of a surface that is not heat-insulated

$$[T(0, \tau) - T_0]_{\tau \to \infty} \to \frac{q\sqrt{a\tau_i}}{\lambda}, \quad (13)$$

which is in good agreement, as follows from the author's data, with an analogous solution but in the assumption of an infinite rate of heat propagation. From this we draw the second conclusion: the effect of a finite rate of heat propagation on the temperature distribution along a rod makes itself felt only during the initial period of the nonsteady segment from $\tau = \tau_d$ to $\tau = \tau^*$. The magnitude of this interval depends on the geometric dimensions and thermophysical properties of the specimen, as well as on the intensity with which heat transfer takes place between the specimen and the ambient medium. However, the magnitude of the delay time τ_d is independent of the heat-transfer intensity.

*As demonstrated to the author by A. V. Luikov and T. L. Perel'man.

When solving the problem of heat propagation in a bounded rod (of length l), to the existing boundary conditions (4)-(6) we should add the condition describing the exchange of heat with the ambient medium at the end of the rod, because now in the place of (5) we should write

$$T(x, 0) = T(l, 0) = T_0 = \text{const.} \tag{5'}$$

Assuming for the sake of simplicity that the heat transfer at the end of the rod is small in comparison with the heat transfer at the side surface, a condition which is well satisfied for thin long specimens ($Q_{face} \approx hQ_{face}/l$), we write the boundary conditions in the form

$$T(l, \tau) = T_0 = \text{const}$$

or

$$\left(\frac{\partial T}{\partial x}\right)_{x=l} = 0. \tag{14}$$

The solution in transformations of the basic equation (12) for conditions (4)-(6) and (14) will be

$$T_L(x, s) - T_0/s = \frac{q \exp(-kx)}{k\lambda s} \frac{1 + \exp[-2k(l-x)]}{1 - \exp(-2kl)}. \tag{15}$$

Expanding

$$\frac{1}{1 - \exp(-2kl)} = 1 + \sum_{n=1}^{\infty} \exp(-2knl)$$

and limiting ourselves to the first term of the series, we find

$$T_L(x, s) - T_0/s = \frac{q}{\lambda ks} \{\exp(-kx) + \exp[-(2l-x)k] + \exp[-(2l+x)k] + \exp[-(4l-x)k]\}, \tag{16}$$

where

$$k = \sqrt{\frac{\tau_r s^2 + s + H}{a}}.$$

Reconverting, we find

$$T(x, \tau) - T_0 = 0 \text{ when } \tau < \tau_{d1} \tag{17}$$

and

$$T(x, \tau) - T_0 = \frac{q}{\lambda}\sqrt{\frac{a}{\tau_r}} \sum_{i=1}^{4} \int_{\tau_{di}}^{\tau} e^{-\frac{t}{2\tau_r}} I_0\left(\frac{\sqrt{1 - 4H\tau_r}\sqrt{t^2 - \tau_{di}^2}}{2\tau_r}\right) dt \tag{18}$$

$$\text{when } \tau > \tau_{di}.$$

For the case of a heat-insulated surface, instead of (18) we have

$$T(x, \tau) - T_0 \cong \frac{q}{\lambda}\sqrt{\frac{a}{\tau_r}} \sum_{i=1}^{4} \int_{\tau_{di}}^{\tau} e^{-\frac{t}{2\tau_r}} I_0\left(\frac{\sqrt{t^2 - \tau_{di}^2}}{2\tau_r}\right) dt. \tag{19}$$

In (18) and (19) the values of τ_{di} are, respectively,

$$\tau_{d1} = x\sqrt{\frac{\tau_r}{a}}; \quad \tau_{d2} = (2l - x)\sqrt{\frac{\tau_r}{a}}; \quad \tau_{d3} = (2l + x)\sqrt{\frac{\tau_r}{a}};$$

$$\tau_{d4} = (4l - x)\sqrt{\frac{\tau_r}{a}}.$$

As we can see, the existence of a surface bounding the length of the rod leads to the appearance of temperature waves, and the change in temperature at a given point is determined by the sum of the solution for an infinite rod and the additional terms characterizing the limited extent of the specimen.

NOTATION

a is the coefficient of thermal diffusivity;
α is the coefficient of heat transfer;
c_p is the specific heat capacity;
ρ is the density;

$h = f/P$	is the form parameter;
f	is the cross-sectional area;
P	is the perimeter;
q	is the specific heat flow;
λ	is the coefficient of thermal conductivity;
T	is the temperature;
T_0	is the temperature of the medium;
τ	is the time;
x	is an instantaneous coordinate;
τ_r	is the relaxation time;
τ_d	is the delay time;
Γ	is the mean free path;
U	is the thermal velocity of the particles;
p	is the pressure;
$\tau_i = c_p\rho h/\alpha$	is the time constant for the process;
l	is the length;
Q	is the quantity of heat.

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